

# How to construct wavelets on local fields of positive characteristic.

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**Abstract:** We present an algorithm for construction step wavelets on local fields of positive characteristic.

**Key words:** Local field, scaling function, wavelets, multiresolution analysis.

## Introduction

In 2004 H.Jiang, D.Li, and N.Jin [10] introduced the notion of multiresolution analysis (MRA) on local fields  $F^{(s)}$  of positive characteristic  $p$ , proved some properties and constructed "Haar MRA" and corresponding "Haar wavelets". The wavelet theory developed in [1, 2, 3, 4, 11]. Construction of non-Haar wavelets is the a basic problem in this theory. The problem of constructing orthogonal MRA on the field  $F^{(1)}$  is studied in detail in the works [6, 7, 8, 12, 16, 17]. S.F.Lukomskii, A.M.Vodolazov [15, 18] considered local field  $F^{(s)}$  as a vector space over the finite field  $GF(p^s)$  and constructed non-Haar wavelets. In [15] the authors construct the mask  $m^{(0)}$  and correspondent refinable function  $\varphi$  using some tree with zero as a root. In this case wavelets  $\Psi = (\psi^{(1)})_{1 \in GF(p^s)}$  may be found from the equality

$$\hat{\psi}^{(1)} = m^{(1)}(\chi)\hat{\varphi}(\chi\mathcal{A}^{-1})$$

where  $\mathcal{A}$  is a dilation operator,  $m^{(1)}(\chi) = m^{(0)}(\chi\mathbf{r}_0^{-1})$ , and  $\mathbf{r}_k^1$  are Rademacher functions. In the article [13], the concept of  $N$ -valid tree was introduced and an algorithm for constructing the mask  $m^{(0)}$  and correspondent refinable function  $\varphi$  was indicated in the field  $F^{(1)}$ . In the articles [14], [5] the mask  $m^{(0)}$  and correspondent refinable function  $\varphi$  were constructed using graph which is obtained from  $N$ -valid tree by adding new arcs. But in this case we cannot define "masks"  $m^{(1)}(\chi)$  by the equation  $m^{(1)}(\chi) = m^{(0)}(\chi\mathbf{r}_0^{-1})$ .

In this article we give an algorithm for construction of "masks"  $m^{(1)}(\chi)$  in general case.

## 1 Basic concepts

Let  $p$  be a prime number,  $s \in \mathbb{N}$ ,  $GF(p^s)$  – finite field. Local field  $F^{(s)}$  of positive characteristic  $p$  is isomorphic (Kovalski-Pontryagin theorem [9]) to the

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set of formal power series

$$a = \sum_{i=k}^{\infty} \mathbf{a}_i t^i, \quad k \in \mathbb{Z}, \quad \mathbf{a}_i \in GF(p^s).$$

Addition and multiplication in the field  $F^{(s)}$  are defined as sum and product of such series. Therefore we will consider local field  $F^{(s)}$  of positive characteristic  $p$  as the field of sequences infinite in both directions

$$a = (\dots, \mathbf{0}_{n-1}, \mathbf{a}_n, \mathbf{a}_{n+1}, \dots), \quad \mathbf{a}_j \in GF(p^s)$$

which have only finite number of elements  $\mathbf{a}_j$  with negative  $j$  nonequal to zero, and the operations of addition and multiplication are defined by equalities

$$\begin{aligned} a \dot{+} b &= ((\mathbf{a}_i \dot{+} \mathbf{b}_i))_{i \in \mathbb{Z}}, \\ ab &= \left( \sum_{i,j:i+j=l} (\mathbf{a}_i \mathbf{b}_j) \right)_{l \in \mathbb{Z}}, \end{aligned} \quad (1)$$

where " $\dot{+}$ " and " $\cdot$ " are respectively addition and multiplication in  $GF(p^s)$ . The norm of the element  $a \in F^{(s)}$  is defined by the equality

$$\|a\| = \|(\dots, \mathbf{0}_{n-1}, \mathbf{a}_n, \mathbf{a}_{n+1}, \dots)\| = \left( \frac{1}{p^s} \right)^n, \quad \text{если } \mathbf{a}_n \neq \mathbf{0}.$$

Therefore

$$F_n^{(s)} = \{a = (\mathbf{a}_j)_{j \in \mathbb{Z}} : \mathbf{a}_j \in GF(p^s); \mathbf{a}_j = \mathbf{0}, \forall j < n\}$$

is a ball of radius  $p^{-ns}$ .

Neighborhoods  $F_n^{(s)}$  are compact subgroups of the group  $F^{(s)+}$ . We will denote them as  $F_n^{(s)+}$ . They have the following properties:

- 1)  $\dots \subset F_1^{(s)+} \subset F_0^{(s)+} \subset F_{-1}^{(s)+} \subset \dots$
- 2)  $F_n^{(s)+} / F_{n+1}^{(s)+} \cong GF(p^s)^+$  и  $\sharp(F_n^{(s)+} / F_{n+1}^{(s)+}) = p^s$ .

It is noted in [15] that the field  $F^{(s)}$  can be described as a linear space over  $GF(p^s)$ . Using this description one may define the multiplication of element  $a \in F^{(s)}$  on element  $\lambda \in GF(p^s)$  coordinatewise, i.e.  $\lambda a = (\dots \mathbf{0}_{n-1}, \lambda \mathbf{a}_n, \lambda \mathbf{a}_{n+1}, \dots)$ , and the modulus  $\lambda \in GF(p^s)$  can be defined as

$$|\lambda| = \begin{cases} 1, & \lambda \neq \mathbf{0}, \\ 0, & \lambda = \mathbf{0}. \end{cases}$$

It is also proved there, that the system  $g_k \in F_k^{(s)} \setminus F_{k+1}^{(s)}$  is a basis in  $F^{(s)}$ , i.e. any element  $a \in F^{(s)}$  can be represented as:

$$a = \sum_{k \in \mathbb{Z}} \lambda_k g_k, \quad \lambda_k \in GF(p^s).$$

From now on we will consider  $g_k = (\dots, \mathbf{0}_{k-1}, (1^{(0)}, 0^{(1)}, \dots, 0^{(s-1)})_k, \mathbf{0}_{k+1}, \dots)$ . In this case  $\lambda_k = \mathbf{a}_k$ . Let us define the sets

$$H_0^{(s)} = \{h \in G : h = \mathbf{a}_{-1}g_{-1} + \mathbf{a}_{-2}g_{-2} + \dots + \mathbf{a}_{-s}g_{-s}\}, s \in \mathbb{N}.$$

$$H_0 = \{h \in G : h = \mathbf{a}_{-1}g_{-1} + \mathbf{a}_{-2}g_{-2} + \dots + \mathbf{a}_{-s}g_{-s}, s \in \mathbb{N}\}.$$

The set  $H_0$  is the set of shifts in  $F^{(s)}$ . It is an analogue of the set of nonnegative integers.

We will denote the collection of all characters of  $F^{(s)+}$  as  $X$ . The set  $X$  generates a commutative group with respect to the multiplication of characters:  $(\chi * \phi)(a) = \chi(a) \cdot \phi(a)$ . Inverse element is defined as  $\chi^{-1}(a) = \overline{\chi(a)}$ , and the neutral element is  $e(a) \equiv 1$ .

Following [15] we define characters  $r_n$  of the group  $F^{(s)+}$  in the following way. Let  $x = (\dots, \mathbf{0}_{k-1}, \mathbf{x}_k, \mathbf{x}_{k+1}, \dots)$ ,  $\mathbf{x}_j = (x_j^{(0)}, x_j^{(1)}, \dots, x_j^{(s-1)}) \in GF(p^s)$ . The element  $\mathbf{x}_j$  can be written in the form  $\mathbf{x}_j = (x_{js+0}, x_{js+1}, \dots, x_{js+(s-1)})$ . In this case

$$x = (\dots, 0, x_{ks+0}, x_{ks+1}, \dots, x_{ks+s-1}, x_{(k+1)s+0}, x_{(k+1)s+1}, \dots, x_{(k+1)s+s-1}, \dots)$$

and the collection of all such sequences  $x$  is Vilenkin group. Thus the equality  $r_n(x) = r_{ks+l}(x) = e^{\frac{2\pi i}{p}(x_{ks+l})}$  defines Rademacher function of  $F^{(s)+}$  and every character  $\chi \in X$  can be described in the following way:

$$\chi = \prod_{n \in \mathbb{Z}} r_n^{a_n}, \quad a_n = \overline{0, p-1}. \quad (2)$$

The equality (2) can be rewritten as

$$\chi = \prod_{k \in \mathbb{Z}} r_{ks+0}^{a_k^{(0)}} r_{ks+1}^{a_k^{(1)}} \dots r_{ks+s-1}^{a_k^{(s-1)}} \quad (3)$$

and let us define

$$r_{ks+0}^{a_k^{(0)}} r_{ks+1}^{a_k^{(1)}} \dots r_{ks+s-1}^{a_k^{(s-1)}} = \mathbf{r}_k^{\mathbf{a}_k}$$

where  $\mathbf{a}_k = (a_k^{(0)}, a_k^{(1)}, \dots, a_k^{(s-1)}) \in GF(p^s)$ . Then (3) takes the form

$$\chi = \prod_{k \in \mathbb{Z}} \mathbf{r}_k^{\mathbf{a}_k}. \quad (4)$$

We will refer to  $\mathbf{r}_k^{(1,0,\dots,0)} = \mathbf{r}_k$  as the Rademacher functions. By definition we set

$$(\mathbf{r}_k^{\mathbf{a}_k})^{\mathbf{b}_k} = \mathbf{r}_k^{\mathbf{a}_k \mathbf{b}_k}, \quad \chi^{\mathbf{b}} = \left( \prod \mathbf{r}_k^{\mathbf{a}_k} \right)^{\mathbf{b}} = \prod \mathbf{r}_k^{\mathbf{a}_k \mathbf{b}}, \quad \mathbf{a}_k, \mathbf{b}_k, \mathbf{b} \in GF(p^s).$$

It follows that if  $\mathbf{x} = ((x_k^{(0)}, x_k^{(1)}, \dots, x_k^{(s-1)}))_{k \in \mathbb{Z}}$  and  $\mathbf{u} = (u^{(0)}, u^{(1)}, \dots, u^{(s-1)}) \in GF(p^s)$  then

$$(\mathbf{r}_k^{\mathbf{u}}, \mathbf{x}) = \prod_{l=0}^{s-1} e^{\frac{2\pi i}{p} u^{(l)} x_k^{(l)}}.$$

In [15] the following properties of characters are proved

- 1)  $\mathbf{r}_k^{\mathbf{u}+\mathbf{v}} = \mathbf{r}_k^{\mathbf{u}} \mathbf{r}_k^{\mathbf{v}}$ ,  $\mathbf{u}, \mathbf{v} \in GF(p^s)$ .
- 2)  $(\mathbf{r}_k^{\mathbf{v}}, \mathbf{u} g_j) = 1$ ,  $\forall k \neq j$ ,  $\mathbf{u}, \mathbf{v} \in GF(p^s)$ .
- 3) The set of characters of the field  $F^{(s)}$  is a linear space  $(X, *, \cdot^{GF(p^s)})$  over the finite field  $GF(p^s)$  with multiplication being an inner operation and the power  $\mathbf{u} \in GF(p^s)$  being an outer operation.
- 4) The set of Rademacher functions  $(\mathbf{r}_k)$  is a basis in the space  $(X, *, \cdot^{GF(p^s)})$ . The dilation operator  $\mathcal{A}$  in local field  $F^{(s)}$  is defined as  $\mathcal{A}x := \sum_{n=-\infty}^{+\infty} \mathbf{a}_n g_{n-1}$ , where  $x = \sum_{n=-\infty}^{+\infty} \mathbf{a}_n g_n \in F^{(s)}$ . In the group of characters it is defined as  $(\chi \mathcal{A}, x) = (\chi, \mathcal{A}x)$ .

## 2 Step Wavelets

We will consider a case of scaling function  $\varphi$ , which generates an orthogonal MRA, being a step function. The set of step functions constant on cosets of a subgroup  $F_M^{(s)}$  with the support  $\text{supp}(\varphi) \subset F_{-N}^{(s)}$  will be denoted as  $\mathfrak{D}_M(F_{-N}^{(s)})$ ,  $M, N \in \mathbb{N}$ . Similarly,  $\mathfrak{D}_{-N}(F_M^{(s)\perp})$  is a set of step functions, constant on the cosets of a subgroup  $F_{-N}^{(s)\perp}$  with the support  $\text{supp}(\varphi) \subset F_M^{(s)\perp}$ .

Let  $\varphi \in \mathfrak{D}_M(F_{-N}^{(s)})$  generate an orthogonal MRA  $\{V_n\}$ , satisfies the refinement equation  $\varphi(x) = \sum_{h \in H_0^{(N+1)}} \beta_h \varphi(\mathcal{A}x - h)$  [15], which we rewrite in a frequency from

$$\hat{\varphi}(\chi) = m^{(0)}(\chi) \hat{\varphi}(\chi \mathcal{A}^{-1}), \quad (5)$$

where

$$m^{(0)}(\chi) = \frac{1}{p^s} \sum_{h \in H_0^{(N+1)}} \beta_h \overline{(\chi \mathcal{A}^{-1}, h)}$$

is the mask of equation (5). There exist methods for constructing  $m^{(0)}(\chi)$  and  $\hat{\varphi}(\chi)$  (see e.g.[5]). We want to construct wavelets  $\psi^{(1)}$ ,  $\mathbf{1} \in GF(p^s)$ ,  $\mathbf{1} \neq \mathbf{0}$  from

refinable function  $\varphi$ . We will find these wavelets  $\psi^{(1)}$  from the equations

$$\hat{\psi}^{(1)}(\chi) = m^{(1)}(\chi)\hat{\varphi}(\chi\mathcal{A}^{-1}),$$

and will call the functions  $m^{(1)}(\chi)$  masks, too. It is evident that  $\hat{\psi}^{(0)}(\chi) = \hat{\varphi}(\chi)$ .

**Theorem 2.1** *Let  $m^{(k)}(\chi)$  ( $k \in GF(p^s)$ ) be a masks that are constant on the cosets of a subgroup  $F_{-N}^{(s)\perp}$  and periodic with any period  $\mathbf{r}_1^{\mathbf{a}_1}\mathbf{r}_2^{\mathbf{a}_2}\dots\mathbf{r}_\nu^{\mathbf{a}_\nu}$ ,  $\mathbf{a}_j \in GF(p^s)$ ,  $\nu \in \mathbb{N}$ . Define wavelets  $\psi^{(1)}$  by the equations*

$$\hat{\psi}^{(1)}(\chi) = m^{(1)}(\chi)\hat{\varphi}(\chi\mathcal{A}^{-1}),$$

where  $\varphi \in \mathfrak{D}_M(F_{-N}^{(s)})$  is a refinable function. The shifts system  $(\psi^{(1)}(x\dot{-}h^{(1)}))$ ,  $\mathbf{l} \in GF(p^s)$ ,  $h^{(1)} \in H_0$  will be orthonormal iff for any  $\mathbf{a}_{-N}\dots\mathbf{a}_{-1} \in GF(p^s)$

$$\sum_{\mathbf{a}_0 \in GF(p^s)} m^{(k)}(F_{-N}^{(s)\perp}\mathbf{r}_{-N}^{\mathbf{a}_{-N}}\dots\mathbf{r}_0^{\mathbf{a}_0})\overline{m^{(l)}(F_{-N}^{(s)\perp}\mathbf{r}_{-N}^{\mathbf{a}_{-N}}\dots\mathbf{r}_0^{\mathbf{a}_0})} = \delta_{k,l}. \quad (6)$$

**Proof. The sufficiency.** Let  $\hat{\varphi}(\chi) \in \mathfrak{D}_{-N}(F_M^{(s)\perp})$  Consider scalar product  $(\varphi(x\dot{-}g), \psi^{(1)}(x\dot{-}h))$ , where  $g, h \in H_0$ .

$$\begin{aligned} (\varphi(x\dot{-}g), \psi^{(1)}(x\dot{-}h)) &= \int_{F^{(s)}} \varphi(x\dot{-}g)\overline{\psi^{(1)}(x\dot{-}h)}d\mu(x) = \\ &= \int_X \hat{\varphi}_{\dot{-}g}(\chi)\overline{\hat{\psi}_{\dot{-}h}^{(1)}(\chi)} = \int_X \hat{\varphi}(\chi)\overline{\hat{\varphi}(\chi\mathcal{A}^{-1})(\chi, g)(\chi, h)\overline{m^{(1)}(\chi)}}d\nu(\chi) = \\ &= \int_{F_M^{(s)\perp}} |\hat{\varphi}(\chi\mathcal{A}^{-1})|^2(\chi, h\dot{-}g)m^{(0)}(\chi)\overline{m^{(1)}(\chi)}d\nu(\chi) = \\ &= |h\dot{-}g = \tilde{h} = \mathbf{h}_{-1}g_{-1}\dot{+}\mathbf{h}_{-2}g_{-2}\dot{+}\dots| = \\ &= \sum_{\mathbf{a}_{-N}, \dots, \mathbf{a}_0, \dots, \mathbf{a}_{M-1}} \int_{F_{-N}^{(s)\perp}\mathbf{r}_{-N}^{\mathbf{a}_{-N}}, \dots, \mathbf{r}_0^{\mathbf{a}_0}, \dots, \mathbf{r}_{M-1}^{\mathbf{a}_{M-1}}} |\hat{\varphi}(F_{-N}^{(s)\perp}\mathbf{r}_{-N}^{\mathbf{a}_{-N}}\dots\mathbf{r}_{M-1}^{\mathbf{a}_{M-1}}\mathcal{A}^{-1})|^2(\chi, \tilde{h})d\nu(\chi) \cdot \\ &\quad \cdot \overline{m^{(0)}(F_{-N}^{(s)\perp}\mathbf{r}_{-N}^{\mathbf{a}_{-N}}\dots\mathbf{r}_0^{\mathbf{a}_0})m^{(1)}(F_{-N}^{(s)\perp}\mathbf{r}_{-N}^{\mathbf{a}_{-N}}\dots\mathbf{r}_0^{\mathbf{a}_0})}d\nu(\chi) = \\ &= \sum_{\mathbf{a}_{-N}, \dots, \mathbf{a}_0} m^{(0)}(F_{-N}^{(s)\perp}\mathbf{r}_{-N}^{\mathbf{a}_{-N}}\dots\mathbf{r}_0^{\mathbf{a}_0})\overline{m^{(1)}(F_{-N}^{(s)\perp}\mathbf{r}_{-N}^{\mathbf{a}_{-N}}\dots\mathbf{r}_0^{\mathbf{a}_0})} \end{aligned}$$

$$\sum_{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{M-1}} |\hat{\varphi}(F_{-N}^{(s)\perp} \mathbf{r}_{-N}^{\mathbf{a}_{-N+1}} \dots \mathbf{r}_0^{\mathbf{a}_1} \dots \mathbf{r}_{M-2}^{\mathbf{a}_{M-1}})|^2 \int_{F_{-N}^{(s)\perp} \mathbf{r}_{-N}^{\mathbf{a}_{-N}} \dots \mathbf{r}_0^{\mathbf{a}_0} \dots \mathbf{r}_{M-1}^{\mathbf{a}_{M-1}}} (\chi, \tilde{h}) d\nu(\chi). \quad (7)$$

By the orthonormality criteria for the system of shifts  $(\varphi(x \dot{-} h))$  of the refinable function  $\varphi \forall \mathbf{a}_{-N}, \dots, \mathbf{a}_0 \in GF(p^s)$  the following equality holds:

$$\sum_{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{M-1}} |\hat{\varphi}(F_{-N}^{(s)\perp} \mathbf{r}_{-N}^{\mathbf{a}_{-N+1}} \dots \mathbf{r}_0^{\mathbf{a}_1} \dots \mathbf{r}_{M-2}^{\mathbf{a}_{M-1}})|^2 = 1.$$

Consider integral from (7)

$$\begin{aligned} \int_{F_{-N}^{(s)\perp} \mathbf{r}_{-N}^{\mathbf{a}_{-N}} \dots \mathbf{r}_0^{\mathbf{a}_0} \dots \mathbf{r}_{M-1}^{\mathbf{a}_{M-1}}} (\chi, \tilde{h}) d\nu(\chi) &= \frac{1}{p^{sN}} \mathbf{1}_{F_{-N}^{(s)\perp}}(\tilde{h}) \mathbf{r}_{-N}^{\mathbf{a}_{-N}}(\tilde{h}) \dots \mathbf{r}_{-1}^{\mathbf{a}_{-1}}(\tilde{h}) = \\ &= \frac{1}{p^{sN}} \mathbf{1}_{F_{-N}^{(s)\perp}}(\tilde{h}) \prod_{j=-N}^{-1} e^{\frac{2\pi i}{p}((\mathbf{h}_j, \mathbf{a}_j))}, \end{aligned}$$

where  $(\mathbf{h}_j, \mathbf{a}_j) = h_j^{(0)} a_j^{(0)} + \dots + h_j^{(s-1)} a_j^{(s-1)}$  is a scalar product.

Let us introduce the following notation:

$$m_{\mathbf{a}_{-N} \dots \mathbf{a}_0}^{(0)} = m^{(0)}(F_{-N}^{(s)\perp} \mathbf{r}_{-N}^{\mathbf{a}_{-N}} \dots \mathbf{r}_0^{\mathbf{a}_0}), \quad m_{\mathbf{a}_{-N} \dots \mathbf{a}_0}^{(1)} = m^{(1)}(F_{-N}^{(s)\perp} \mathbf{r}_{-N}^{\mathbf{a}_{-N}} \dots \mathbf{r}_0^{\mathbf{a}_0}).$$

Then we obtain

$$\begin{aligned} &(\varphi(\cdot \dot{-} g), \psi^{(1)}(\cdot \dot{-} h)) = \\ &= \frac{1}{p^{sN}} \mathbf{1}_{F_{-N}^{(s)\perp}}(\tilde{h}) \sum_{\mathbf{a}_{-N}, \dots, \mathbf{a}_0} m_{\mathbf{a}_{-N} \dots \mathbf{a}_0}^{(0)} \overline{m_{\mathbf{a}_{-N} \dots \mathbf{a}_0}^{(1)}} \prod_{j=-N}^{-1} e^{\frac{2\pi i}{p}((\mathbf{h}_j, \mathbf{a}_j))} = \\ &= \begin{cases} 0 & \text{if } \tilde{h} \notin F_{-N}^{(s)\perp}; \\ \frac{1}{p^{sN}} \sum_{\mathbf{a}_{-N}, \dots, \mathbf{a}_0} m_{\mathbf{a}_{-N} \dots \mathbf{a}_0}^{(0)} \overline{m_{\mathbf{a}_{-N} \dots \mathbf{a}_0}^{(1)}} & \text{if } \tilde{h} = 0; \\ \frac{1}{p^{sN}} \sum_{\mathbf{a}_{-N}, \dots, \mathbf{a}_{-1}} \prod_{j=-N}^{-1} e^{\frac{2\pi i}{p}((\mathbf{h}_j, \mathbf{a}_j))} \sum_{\mathbf{a}_0} m_{\mathbf{a}_{-N} \dots \mathbf{a}_0}^{(0)} \overline{m_{\mathbf{a}_{-N} \dots \mathbf{a}_0}^{(1)}} & \text{if } \tilde{h} \neq 0, \tilde{h} \in F_{-N}^{(s)\perp}. \end{cases} \quad (8) \end{aligned}$$

For  $(\psi^{(\mathbf{k})}(x \dot{-} g), \psi^{(1)}(x \dot{-} h))$  we can derive similar equality:

$$\begin{aligned} &(\psi^{(\mathbf{k})}(\cdot \dot{-} g), \psi^{(1)}(\cdot \dot{-} h)) = \\ &= \frac{1}{p^{sN}} \mathbf{1}_{F_{-N}^{(s)\perp}}(\tilde{h}) \sum_{\mathbf{a}_{-N}, \dots, \mathbf{a}_0} m_{\mathbf{a}_{-N} \dots \mathbf{a}_0}^{(\mathbf{k})} \overline{m_{\mathbf{a}_{-N} \dots \mathbf{a}_0}^{(1)}} \prod_{j=-N}^{-1} e^{\frac{2\pi i}{p}((\mathbf{h}_j, \mathbf{a}_j))} \end{aligned}$$

$$= \begin{cases} 0 & \text{if } \tilde{h} \notin F_{-N}^{(s)\perp}; \\ \frac{1}{p^{sN}} \sum_{\mathbf{a}_{-N}, \dots, \mathbf{a}_0} m_{\mathbf{a}_{-N} \dots \mathbf{a}_0}^{(\mathbf{k})} \overline{m_{\mathbf{a}_{-N} \dots \mathbf{a}_0}^{(\mathbf{l})}} & \text{if } \tilde{h} = 0; \\ \frac{1}{p^{sN}} \sum_{\mathbf{a}_{-N}, \dots, \mathbf{a}_{-1}} \prod_{j=-N}^{-1} e^{\frac{2\pi i}{p}((\mathbf{h}_j, \mathbf{a}_j))} \sum_{\mathbf{a}_0} m_{\mathbf{a}_{-N} \dots \mathbf{a}_0}^{(\mathbf{k})} \overline{m_{\mathbf{a}_{-N} \dots \mathbf{a}_0}^{(\mathbf{l})}} & \text{if } \tilde{h} \neq 0, \tilde{h} \in F_{-N}^{(s)\perp}. \end{cases} \quad (9)$$

Thus, if masks  $m^{(\mathbf{j})}$  for all  $\mathbf{a}_{-N} \dots \mathbf{a}_{-1} \in GF(p^s)$  satisfy the condition

$$\sum_{\mathbf{a}_0} m_{\mathbf{a}_{-N} \dots \mathbf{a}_0}^{(\mathbf{k})} \overline{m_{\mathbf{a}_{-N} \dots \mathbf{a}_0}^{(\mathbf{l})}} = \delta_{\mathbf{k}, \mathbf{l}},$$

then the system of shifts  $(\psi^{(\mathbf{l})}(x \dot{-} h^{(\mathbf{l})}))$ ,  $\mathbf{l} \in GF(p^s)$  is an orthonormal system. **The necessity.** Let us fix  $\mathbf{k}, \mathbf{l} \in FG(p^s)$  and consider equalities (8), (9) as a system of linear equation with unknowns  $x_{\mathbf{a}_{-N} \dots \mathbf{a}_{-1}}^{\mathbf{k}, \mathbf{l}} = \sum_{\mathbf{a}_0} m_{\mathbf{a}_{-N} \dots \mathbf{a}_0}^{(\mathbf{k})} \overline{m_{\mathbf{a}_{-N} \dots \mathbf{a}_0}^{(\mathbf{l})}}$  and consider the matrix  $A$  of this system.

It is obvious that  $A$  is a square matrix  $p^{sN} \times p^{sN}$ . Let us prove that its determinant is nonequal to zero.

Let us start with  $N = 1, s = 1$ . In this case

$$A = \frac{1}{p} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & e^{\frac{2\pi i}{p}} & e^{\frac{2\pi i}{p} \cdot 2} & \dots & e^{\frac{2\pi i}{p} \cdot (p-1)} \\ 1 & e^{\frac{2\pi i}{p} \cdot 2} & e^{\frac{2\pi i}{p} \cdot 2 \cdot 2} & \dots & e^{\frac{2\pi i}{p} \cdot 2 \cdot (p-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & e^{\frac{2\pi i}{p} \cdot (p-1)} & e^{\frac{2\pi i}{p} \cdot (p-1) \cdot 2} & \dots & e^{\frac{2\pi i}{p} \cdot (p-1) \cdot (p-1)} \end{pmatrix} = V, \quad (10)$$

where  $V$  is Vandermonde matrix, which is known to have nonzero determinant.

For the sake of clarity let us consider a case  $N = 2, s = 1$ . In this case the matrix  $A$  may be represented as block matrix

$$A = \frac{1}{p} \begin{pmatrix} V & V & V & \dots & V \\ V & e^{\frac{2\pi i}{p}} V & e^{\frac{2\pi i}{p} \cdot 2} V & \dots & e^{\frac{2\pi i}{p} \cdot (p-1)} V \\ V & e^{\frac{2\pi i}{p} \cdot 2} V & e^{\frac{2\pi i}{p} \cdot 2 \cdot 2} V & \dots & e^{\frac{2\pi i}{p} \cdot 2 \cdot (p-1)} V \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ V & e^{\frac{2\pi i}{p} \cdot (p-1)} V & e^{\frac{2\pi i}{p} \cdot (p-1) \cdot 2} V & \dots & e^{\frac{2\pi i}{p} \cdot (p-1) \cdot (p-1)} V \end{pmatrix} = V \otimes V, \quad (11)$$

where  $\otimes$  symbol corresponds to Kronecker product. By the properties of Kronecker product  $\det V \otimes V = (\det V)^p (\det V)^p = (\det V)^{2p} \neq 0$ . Thus, again matrix  $A$  is nonsingular.

For the case of arbitrary  $N$ ,  $s = 1$  matrix  $A$  can be represented as  $A = V \otimes V \otimes \dots \otimes V$   $N$  times and will again have nonzero determinant by the properties of Kronecker product.

Similarly, when  $N$  and  $s$  are both arbitrary  $A = V \otimes V \otimes \dots \otimes V$   $sN$  times. Thus, the system is nonsingular and has a unique solution, which proves the necessity.  $\square$

Theorem 2.1 can be reformulated in the following way:  $m^{(\mathbf{k})}(\chi)$  are the masks of corresponding step compactly supported orthonormal wavelets  $\psi^{(\mathbf{l})}(\chi)$  if and only if for each  $\mathbf{a}_{-N} \dots \mathbf{a}_{-1} \in GF(p^s)$  matrix  $M(\mathbf{a}_{-N} \dots \mathbf{a}_{-1})$  with elements

$$M_{\mathbf{l}, \mathbf{a}_0}(\mathbf{a}_{-N}, \dots, \mathbf{a}_{-1}) = m^{(\mathbf{l})}(F_{-N}^{(s)\perp} \mathbf{r}_{-N}^{\mathbf{a}_{-N}} \dots \mathbf{r}_0^{\mathbf{a}_0})$$

is unitary. The sufficiency of this theorem was proved in [10] (theorem 3). For step refinable functions the condition (6) is necessary and sufficient. If the condition (6) is fulfilled then the functions  $\hat{\psi}^{(\mathbf{l})}(\chi) = m^{(\mathbf{l})}(\chi) \hat{\varphi}(\chi \mathcal{A}^{-1})$  form a wavelet system [10]. For a step refinable function we can describe an algorithm for constructing masks  $m^{(\mathbf{l})}$  and wavelets  $\psi^{(\mathbf{l})}$ ,  $\mathbf{l} \in GF(p^s)$ .

Let us assume we have all the values of  $m^{(0)}(\chi)$ . We may obtain them using an algorithm presented in [5]. Recall the notation:

$$m_{\mathbf{a}_{-N} \dots \mathbf{a}_0}^{(0)} = m^{(0)}(F_{-N}^{(s)\perp} \mathbf{r}_{-N}^{\mathbf{a}_{-N}} \dots \mathbf{r}_0^{\mathbf{a}_0}), \quad m_{\mathbf{a}_{-N} \dots \mathbf{a}_0}^{(\mathbf{l})} = m^{(\mathbf{l})}(F_{-N}^{(s)\perp} \mathbf{r}_{-N}^{\mathbf{a}_{-N}} \dots \mathbf{r}_0^{\mathbf{a}_0}).$$

**1)** For each  $\mathbf{a}_{-N} \dots \mathbf{a}_{-1}$  we construct a matrix  $M(\mathbf{a}_{-N} \dots \mathbf{a}_{-1}) \in Mat_{p^s \times p^s}(\mathbb{C})$  with elements  $M_{\mathbf{l}, \mathbf{a}_0}(\mathbf{a}_{-N} \dots \mathbf{a}_{-1})$  the following way. The first row consists of all the values

$$m_{\mathbf{a}_{-N} \dots \mathbf{a}_{-1}, 0}^{(0)}, m_{\mathbf{a}_{-N} \dots \mathbf{a}_{-1}, 1}^{(0)}, \dots, m_{\mathbf{a}_{-N} \dots \mathbf{a}_{-1}, p^{s-1}}^{(0)}$$

where  $\mathbf{a}_{-N} \dots \mathbf{a}_{-1}$  are fixed and  $j = a_0^{(0)} + a_0^{(1)}p + \dots + a_0^{(s-1)}p^{s-1}$  calculated from  $\mathbf{a}_0 = (a_0^{(0)}, a_0^{(1)}, \dots, a_0^{(s-1)})$ . Supplement this matrix to unitary in the following way.

If  $m_{\mathbf{a}_{-N} \dots \mathbf{a}_{-1}, 0}^{(0)} \neq 0$  then we make  $M_{\mathbf{l}, \mathbf{l}} = 1$  for  $\mathbf{l} \neq \mathbf{0}$  and  $M_{\mathbf{l}, \mathbf{a}_0} = 0$  for  $\mathbf{l} \neq \mathbf{0}, \mathbf{l} \neq \mathbf{a}_0$ .

If  $m_{\mathbf{a}_{-N} \dots \mathbf{a}_{-1}, 0}^{(0)} = 0$  then there exists number

$$j = j(\mathbf{a}_0) = a_0^{(0)} + a_0^{(1)}p + \dots + a_0^{(s-1)}p^{s-1}$$

for which  $m_{\mathbf{a}_{-N} \dots \mathbf{a}_{-1}, j}^{(0)} \neq 0$ . This nonzero value exists by the property of  $m^{(0)}$  (see e.g. [10]). In this case we make  $M_{\mathbf{j}, \mathbf{j}} = 1$ ,  $M_{\mathbf{l}, \mathbf{l}} = 1$  for  $\mathbf{l} \neq \mathbf{0}, \mathbf{l} \neq \mathbf{j}$ , and  $M_{\mathbf{l}, \mathbf{a}_0} = 0$  in another case.

**2)** Run the Gram-Schmidt process on each matrix in order to make them unitary.



3) Now for each  $\mathbf{l} \in GF(p^s)$ ,  $\mathbf{l} \neq \mathbf{0}$  we find the values of the mask  $m^{(\mathbf{l})}$  from the equalities

$$m^{(\mathbf{l})}(F_{-N}^{(s)\perp} \mathbf{r}_{-N}^{\mathbf{a}_{-N}} \dots \mathbf{r}_0^{\mathbf{a}_0}) = M_{\mathbf{l}, \mathbf{a}_0}(\mathbf{a}_{-N} \dots \mathbf{a}_{-1}).$$

4) The wavelets  $\psi^{(\mathbf{l})}$  can be obtained using the formula

$$\hat{\psi}^{(\mathbf{l})}(\chi) = m^{(\mathbf{l})}(\chi) \hat{\varphi}(\chi \mathcal{A}^{-1})$$

and performing inverse Fourier transform.

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